Case II – $D = 0$ i.e one root of Discriminating cube is zero

$$r = ul_3 + vm_3 + wn_3 \neq 0$$

where l_3, m_3, n_3 are actual direction cosines of axis corresponding to $\lambda_3 = 0$

In this case, $f(x,y,z) = 0$ reduces either of the following two forms:

$Ax^2 + By^2 + Cz = 0$ → Elliptical Paraboloid

$Ax^2 - By^2 + Cz = 0$ → Hyperbolic paraboloid

Ex – Reduce the equation

$$6y^2 - 18yz - 6zx + 2xy - 9x + 5y - 5z + 2 = 0$$

to the standard form, and state nature of the the surface represented by it.

$\Rightarrow$ The given equations is

$$f(x,y,z) \equiv 6y^2 - 18yz - 6zx + 2xy - 9x + 5y - 5z + 2 = 0 \quad \text{(1)}$$

comparing it with

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

$a = 0$ $\quad f = -9$ $\quad u = \frac{-9}{2}$ $\quad d = 2$

$b = 6$ $\quad g = -3$ $\quad v = \frac{5}{2}$

$c = 0$ $\quad h = 1$ $\quad w = -\frac{5}{2}$

$A = bc - f^2$

$= 6(0) - (-9)^2$

$= -81$

$D = abc + 2fgh - af^2 - bg^2 - ch^2$

$= (0)(6)(0) + 2(-9)(-3)(1) - 0 - 6(-3)^2 - 0$

$= 0 + 54 - 0 - 54 - 0 = 0$

$B = ca - g^2$

$= (0)(0) - (-3)^2$

$= -9$

$C = ab - h^2$

$= 0(6) - (1)^2$

$= -1$

The discriminating cubic is.

$$\lambda^3 - (a+b+c)\lambda^2 + (A+B+C)\lambda - D = 0$$

$$\lambda^3 - 6\lambda^2 + (-81 - 9 - 1)\lambda - 0 = 0$$

$$\lambda^3 - 6\lambda^2 - 91\lambda = 0$$

$$\lambda(\lambda^2 - 6\lambda - 91) = 0$$

$$\lambda(\lambda+7)(\lambda-13) = 0$$

$$\lambda = 0, -7, 13$$

Take $\lambda_1 = -7$, $\lambda_2 = 13$, $\lambda_3 = 0$

The principal direction corresponding to $\lambda = 0$ is given by-

$\Rightarrow$ $al + hm + gn = 0$

$0 + 1(m) - 3m = 0$

$m - 3n = 0$ ——— (2)

$\Rightarrow$ $hl + bm + fn = 0$

$1.(l) + 6m - 9n = 0$

$l + 6m - 9n = 0$ ——— (3)

$\Rightarrow$ $gl + fm + cn = 0$

$$-3l + (-9)m + 0 = 0$$

$$-3l - 9m = 0$$

$$3l + 9m = 0 \quad \text{———} \; (4)$$

Solving (1) and (3) we get

By using cross multiplication method.

$$\frac{l}{-9+18} = \frac{m}{-3+0} = \frac{n}{0-1}$$

$$\frac{l}{9} = \frac{m}{-3} = \frac{n}{-1}$$

$$\frac{l}{-9} = \frac{m}{3} = \frac{n}{1} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{81+9+1}}$$

$$= \frac{1}{\sqrt{91}}$$

$$l = \frac{-9}{\sqrt{91}}, \quad m = \frac{3}{\sqrt{91}}, \quad n = \frac{1}{\sqrt{91}}$$

Now, $r = ul + vm + wn$

$$= -\frac{9}{2}\left(-\frac{9}{\sqrt{91}}\right) + \frac{5}{2} \cdot \frac{3}{\sqrt{91}} - \frac{5}{2} \cdot \frac{1}{\sqrt{91}}$$

$$= \frac{1}{2} \cdot \sqrt{91} \neq 0$$

The Reduced eqn is

$$\lambda_1 x^2 + \lambda_2 y^2 + 2nz = 0$$

$$-7x^2 + 13y^2 + \sqrt{91} \cdot z = 0$$

$$7x^2 - 13y^2 = \sqrt{91} \cdot z$$

which is hyperbolic paraboloid.